# Physics-Based High-Efficiency Analysis of Membranophones Using a Spectral Method

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## ABSTRACT

We propose a high-efficiency analysis method of membranophones such as drums or timpani. The main strategies to make the computation more efficient are as follows. Since most of membranophones have axisymmetric shapes, they can be analyzed theoretically in the circumferential direction by using Fourier series expansions. This strategy can reduce the spatial dimensions to be discretized. Then, the rotating section is analyzed by using spectral methods, which are high-accuracy numerical methods. Fourier series expansions and spectral methods in the analyses of membranophones are formulated for each component of the membranophones. Although spectral methods have restrictions concerned with applicable shapes and boundary conditions, applying several techniques enables the methods to deal with complex shapes and boundary conditions. Components of a membranophone are coupled through coupling conditions of pressure and velocity. It is shown that the proposed method can analyze membranophones accurately with a small number of degrees of freedom and low memory requirement.

## 1. INTRODUCTION

Sound analysis and synthesis of musical instruments are useful for their acoustic design and playing virtual instruments. In particular, physics-based synthesis has been studied and developed actively in recent years. It can reflect changes of physical parameters such as properties of materials or excitation conditions to timbres[1]. That is a difficult task with synthesis based on signal processing. However, physics-based synthesis requires much computational cost for spatial discretization in general numerical calculations. Thus, most of studies concerned with it focused on implementation of computational methods on hardware such as GPU or FPGA rather than the methods themselves[2, 3].

This study deals with membranophones such as drums and timpani, and proposes an efficient computational method of them. In general, most of membranophones have axisymmetric shapes. Therefore, their vibration displacements and acoustic velocity potentials can be expanded in Fourier series in the circumferential direction. It means that it is unnecessary to discretize the circumferential direction, thereby, the spatial dimension to be discretized can be reduced.

The rotating section, which generates the total axisymmetric shape, is analyzed by using spectral methods, which are high-accuracy numerical methods. Global interpolation functions used in spectral methods enable the methods to offer greater accuracy than other numerical methods such as finite difference methods (FDMs) or finite element methods (FEMs), which use local interpolation functions. In other words, when only lower accuracy is required, spectral methods demand less computational cost than the others. We have formulated spectral methods for vibration analyses of a stretched circular plate and cylindrical shell, and shown that they could analyze the vibration fields more accurately with a small number of degrees of freedom and less memory than FEMs in our previous studies[4, 5].

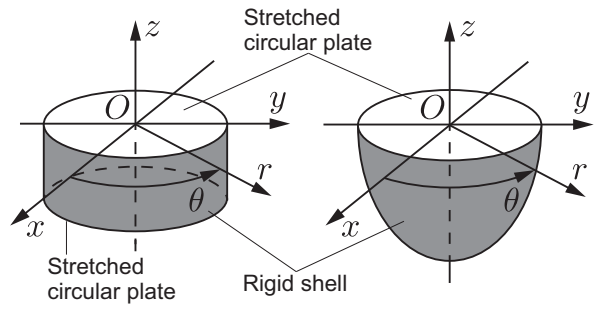


**Figure 1**. Analysis models of a drum (left) and timpani (right).

Although spectral methods have restrictions concerned with applicable shapes and boundary conditions, applying several techniques enables the methods to deal with complex shapes and boundary conditions. After applying Fourier series and spectral methods to each component of membranophones, obtained matrix equations are coupled through coupling conditions of velocity and pressure.

Finally, calculated and measured frequency response functions are compared and it is shown that the proposed method is effective as an analysis method of membranophones.

## 2. PHYSICAL MODEL

Analysis models are shown in Fig. 1. Although the shells slightly vibrate in a real situation, they are considered rigid for ease in this study.

### 2.1. Stretched Circular Plate

A head of a membranophone is modeled as a stretched circular plate since it has both tension and bending stiffness. The

vibration equation of the stretched plate is expressed as

$$(B\nabla^4 - T\nabla^2)(\zeta + \beta\frac{\partial\zeta}{\partial t}) + \rho_{\mathrm{p}}\frac{\partial^2\zeta}{\partial t^2} = f + p, \qquad (1)$$

where $\zeta$, $B$, $T$, $\rho_{\mathrm{p}}$, $f$ and $p$ are vibration displacement, bending stiffness, tension, surface density, mechanical excitation pressure and sound pressure, respectively. $\beta$ is a coefficient determining damping effect proportional to stiffness.

### 2.2. Sound Field

The wave equation governing sound fields inside and outside a membranophone cavity is expressed as

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \qquad (2)$$

where $\phi$ and $c$ are acoustic velocity potential and sound velocity, respectively. Sound pressure $p$ is given by $p = \rho_{\mathrm{a}}(\partial\phi/\partial t)$, where $\rho_{\mathrm{a}}$ is the density of air.

## 3. ANALYSIS METHOD

### 3.1. Fourier Series Expansion

The physical models shown in Fig. 1 have axisymmetric shapes. Therefore vibration displacement and velocity potential can be expanded in the circumferential direction as follows

$$\zeta(r,\theta,t) = \sum_{n=-\infty}^{\infty} \zeta_n(r,t)e^{jn\theta}, \qquad (3)$$

$$\phi(r,\theta,z,t) = \sum_{n=-\infty}^{\infty} \phi_n(r,z,t)e^{jn\theta}. \qquad (4)$$

In practice, the above infinite series has to be truncated at a finite order. Substituting the displacement and velocity potential into the differential equations in the previous section and using the orthogonality of trigonometric functions, we obtain decoupled differential equations with respect to only $r$, $z$ and $t$; the equations do not include differentiation with respect to $\theta$.

### 3.2. Spectral Method

The differential equations obtained in the previous section are analyzed numerically by using spectral methods with only discretizing the $rz$-plane. Spectral methods are some versions of weighted residual methods and classified according to the type of interpolation function and weighting function. In this study, spectral nodal Galerkin methods[6], which use Lagrange basis functions spreading over the whole analyzed domain as both interpolation functions and weighting functions, are adopted. Although spectral methods can offer greater accuracy than FDMs or FEMs, they have drawbacks concerned with applicable shapes and boundary conditions. Some special treatments are introduced to overcome the drawbacks.

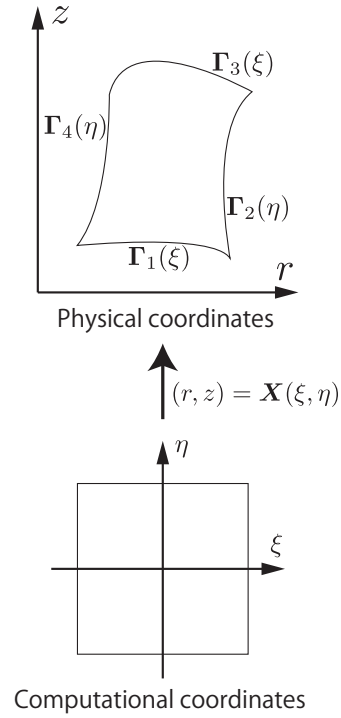


**Figure 2**. A map between the physical space coordinates $(r, z)$ and computational space coordinates $(\xi, \eta)$.

#### 3.2.1. *Plate vibration field*

In a typical spectral method with Lagrange type of differentiation matrices, only one equation per one unknown is obtained. However, for the plate vibration field, two boundary conditions have to be imposed at a boundary node because the vibration equation of the stretched plate is a fourth-order differential equation.

Use of Hermite interpolation type of differentiation matrices can overcome the problem[5]. The differentiation matrices are derived by using Hermite interpolation that interpolates with nodal derivatives in addition to nodal values instead of Lagrange interpolation. This treatment enables two boundary conditions to be imposed at the boundary node.

Discretizing the differential equation of the plate vibration field according to spectral nodal Galerkin formulation, we obtain the same form of matrix equation as one in an FEM,

$$\boldsymbol{M}^{\mathrm{p}}\ddot{\boldsymbol{\zeta}}_n + \boldsymbol{C}_n^{\mathrm{p}}\dot{\boldsymbol{\zeta}}_n + \boldsymbol{K}_n^{\mathrm{p}}\boldsymbol{\zeta}_n = \boldsymbol{f}_n + \boldsymbol{p}_n, \qquad (5)$$

where $\boldsymbol{M}^{\mathrm{p}}$, $\boldsymbol{C}_n^{\mathrm{p}}$, $\boldsymbol{K}_n^{\mathrm{p}}$, $\boldsymbol{\zeta}_n$, $\boldsymbol{f}_n$ and $\boldsymbol{p}_n$ are the mass matrix, damping matrix, stiffness matrix, vibration displacement vector, mechanical excitation force vector and sound pressure vector, respectively. The subscript $n$ represents an order of the Fourier series. The entries of the damping and stiffness matrices depend on the order $n$ while those of the mass matrix are independent of it.

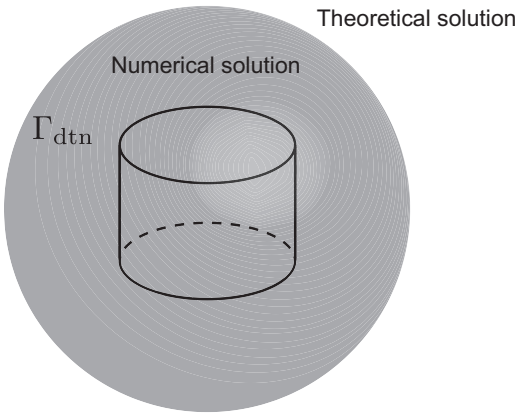


**Figure 3**. A spherical artificial boundary around an obstacle for a DtN method.

### 3.2.2. *Sound field*

Spectral methods can ordinarily analyze only regular-shaped domains due to their global interpolation functions. However, the shapes of the sound fields inside and outside a membranophone cavity are not always regular. Thus, we introduce generalized curvilinear coordinates $(\xi, \eta)$ in order to analyze such an irregular-shaped domain.

Transform of Eq. 2 with a map $\boldsymbol{X}(\xi, \eta)$ between the physical coordinates $(r, z)$ and computational coordinates $(\xi, \eta)$ shown in Fig. 2 yields the wave equation in the generalized curvilinear coordinates. Discretizing the wave equation as in the case of the plate vibration field, we obtain

$$\boldsymbol{M}^{\mathrm{a}}\ddot{\boldsymbol{\phi}}_n + \boldsymbol{K}_n^{\mathrm{a}}\boldsymbol{\phi}_n = \boldsymbol{v}_n, \tag{6}$$

where $\boldsymbol{M}^{\mathrm{a}}$, $\boldsymbol{K}_n^{\mathrm{a}}$, $\boldsymbol{\phi}_n$ and $\boldsymbol{v}_n$ are the mass matrix, stiffness matrix, velocity potential vector and acoustical particle velocity vector, respectively.

### 3.2.3. *Boundary and coupling conditions*

We discuss boundary and coupling conditions here. First, the shell or kettle of a membranophone is considered acoustically rigid. Then acoustical particle velocity has to be zero on it.

Next, although the sound field outside a membranophone cavity is an infinite domain, it has to be truncated at a finite domain and a proper boundary condition has to be imposed at the artificial boundary when it is analyzed with spectral methods. In this study, a Dirichlet-to-Neumann (DtN) method[7] is adopted. It is the method that introduces a spherical artificial boundary $\Gamma_{\mathrm{dtn}}$ around an obstacle and couples a numerical solution with a theoretical solution at the boundary (Fig. 3). Since it expresses an exact non-reflecting boundary condition, one can make a domain which is numerically analyzed smaller than other approximated non-reflecting boundary conditions.

Finally, the plate vibration field and sound fields inside and outside the cavity are coupled through coupling conditions of pressure and velocity. The sound pressure vector to the plate $\boldsymbol{p}_n$ is given by the difference between the interior and exterior sound pressure $\rho_{\mathrm{a}}\dot{\boldsymbol{\phi}}_n$. The acoustical particle velocity vectors to the sound fields $\boldsymbol{v}_n$ are given by the vibration velocity vector $\dot{\boldsymbol{\zeta}}_n$.The coupling conditions yields a coupled matrix equation. Solving the equation and substituting $\zeta_n$ and $\phi_n$ into Eqs. (3) and (4), we obtain vibration displacement and acoustic velocity potential at arbitrary points.

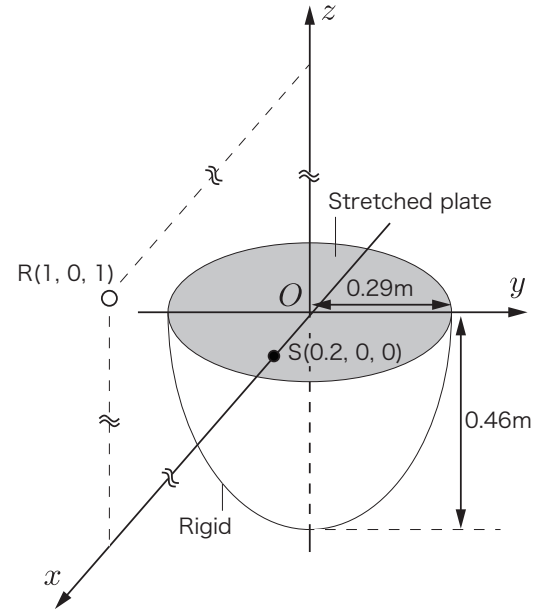


**Figure 4**. The size of the measured and calculated timpani drum and positions of excitation point and receiving point.

### 3.2.4. *Analysis of temporal differentiation*

Equations (5) and (6) include temporal differentiation and they can be solved by using time integration schemes. However, they need verification concerned with the stability. Therefore, at a stage prior to the analysis in the time domain, we let the time factor be $e^{j\omega t}$ and solve the equations in the frequency domain in this study.

## 4. NUMERICAL RESULTS

In order to validate the proposed method, frequency response functions of a timpani drum were measured and calculated.

### 4.1. Measurement and Calculation Condition

Figure 4 shows the size of a measured and calculated timpani drum and the positions of a excitation point S and receiving point R. The parameter values were the following: $T = 5800\ \mathrm{N/m}$, $B = 1.83 \times 10^{-4}\ \mathrm{Pa \cdot m^4}$, $\rho_{\mathrm{p}} = 0.25\ \mathrm{kg/m^2}$, $\beta = 1.0 \times 10^{-6}\ \mathrm{s}$, $c = 344\ \mathrm{m/s}$ and $\rho_{\mathrm{a}} = 1.21\ \mathrm{kg/m^3}$.

In measurement, the timpani head was excited impulsively by using a drum stick and a frequency response function of radiated sound was measured with an FFT analyzer.

In calculation, the artificial boundary to which a DtN map was applied was set on a sphere with a radius of 0.5 m and

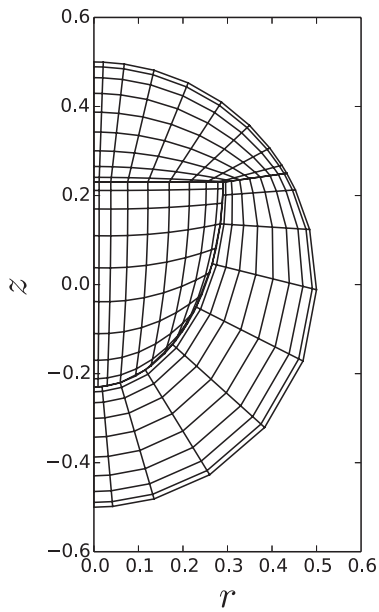


**Figure 5**. The nodes of the calculated timpani drum.

the $rz$-plane was discretized as shown in Fig. 5. The degree of freedom of the coupled matrix equation to be solved was 310 and the Fourier series in the circumferential direction was truncated at the 10th order.

### 4.2. Comparison Between Measurement and Calculation

Calculated and measured frequency response functions are shown in Fig. 6. The blue line and red one represent the calculated frequency response function and measured one, respectively.

The level differences of the first and fourth peaks, which correspond to (0, 1) and (0, 2) mode, are seen. It is possible that the differences are attributed to disagreement of parameter values or excitation force. However, the other peaks show near levels and both the results show good agreement in terms of eigenfrequencies. Therefore, the result suggests that the proposed method be valid as an analysis method of a membranophone.

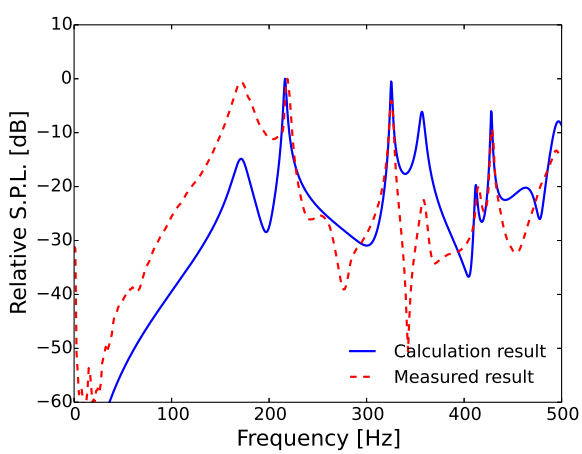


**Figure 6**. Comparison of calculation and measured results.

## 5. CONCLUSIONS

A high-efficiency analysis method of membranophones was proposed. It was shown that a frequency response function of a timpani drum calculated with the proposed method showed good agreement with measured ones. The proposed method is based on Fourier series expansions and spectral methods as well as proposed methods in our previous studies[4, 5]. From results in the studies, it is probable that the proposed method in this study could also significantly reduce the degree of freedom necessary for computation and analyze membranophones more efficiently than FEMs.

Although matrix equations were solved in the frequency domain in this study, time integration schemes such as a central difference method or Newmark-$\beta$ method also could be applied. In particular, since mass matrices in spectral methods are diagonal matrices, explicit schemes, which have a problem of stability, would be effective and could reduce computational cost. Analysis in the time domain is our future work.

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